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Advanced-Level Students' Understanding of Exponential Equations: A Zimbabwean Case Study

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Abstract
This paper reports on a study which explored high school students' conceptual understanding of the techniques of exponential functions. Thirty-one advanced-level students participated in the study. The study used APOS (Action-Process-Object-Schema) theory, a constructivist theory framework, to investigate participants' conceptual understanding of exponential functions. Activity sheets constructed with tasks based on exponential equations were administered to the participants. The written responses were used to identify participants' mental constructions of these concepts. Furthermore, interviews were carried out to clarify participants' written responses. The written responses and interview discussions pointed out that participants exhibited procedural tendencies in exponential functions. Most of the participants could not solve exponential equations, especially the radioactive-decay functions. In addition, many participants did not have appropriate mental constructions at the process, object and schema levels, since most of them could not coordinate processes and encapsulate them into an object. This paper raises some important implications for mathematics education and further provides applications of genetic decomposition design and modification.

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INTRODUCTION

Students need a basic knowledge of mathematics to extend their learning to a higher level (Maat & Zakaria, 2010). This discipline is actually called for by almost every sector in the world. Fauzi and Priatna (2019) asserted that algebra is a mathematical topic that is vital in high school mathematics curricula. However, research has shown that many students around the world show a poor understanding of the algebraic oncepts (Jupri et.al, 2021; Makgakga & Sepeng, 2013; Wahyuni & Angraini, 2019).

Understanding is vital in the teaching of mathematics. According to the National Council of Teachers of Mathematics (NCTM, 2000), "students must learn mathematics with understanding" (p. 20). The goal of mathematics teaching is for students to understand the mathematics concepts presented to them. Perkins and Blythe (1994) defined understanding as the ability to explain and justify, find evidence and examples of, generalize, apply, and represent a topic in a new way. Skemp (1976) distinguished between two different types of understanding in mathematics, which he termed instrumental understanding and relational understanding. Other researchers have used the term procedural understanding to

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refer to instrumental understanding and conceptual understanding for relational understanding (Hiebert, 2013; Hiebert & Lefevre, 1986; Van de Walle, 2001). Hiebert (2013) defined procedural understanding as the ability to solve problems in a step-by-step manner, logically and with deterministic instructions for how to solve a problem, whereas Skemp (1976) defined relational understanding as "[k]nowing both what to do and why" (p. 20). Hiebert (2013) defined conceptual understanding as knowledge that is rich in relationships.

Exponential equations are a vital part of high school mathematics. It is an important branch of mathematics which cuts across all spheres. From a contemporary perspective, mathematical functions are important in school mathematics curricula because they serve as a bridge between mathematical topics such as equations, functions and polynomials and should be taught with understanding (Curriculum and Assessment Policy Statement, 2012). In Zimbabwe, learners are introduced to the concept of exponential equations when they are in form 1. The concept is continuously developed up to ordinary level, advanced level and then at tertiary level, but students often find it difficult. However, the difficulty may not be merely because of the content but also because of the transition from elementary level.

The above facts and the concerns about the high mathematics failure rate in Zimbabwean secondary schools have prompted us to conduct this study to explore if students' understanding of exponential equations concepts could be the cause. Also, in the Zimbabwean context, emphasis is placed more on passing the national examinations than on conceptual understanding. Dubinsky (1991) believed that mathematics teachers should work hard to help learners construct schemas so that they can understand mathematical concepts. It is the goal of this study to gain more insight about how learners understand the exponential equation concept. The study explores conceptual understanding using the APOS levels exhibited by students and suggests ways of organizing content for better understanding of exponential equations, thus producing suitable genetic decompositions for the concept. More specifically, APOS theory through the genetic decomposition could lead us towards pedagogical strategies that in turn lead to a marked improvement in the understanding of the methods of solving exponential equations, through genetic decomposition. The study is guided by the following questions:

a. What are the challenges faced by advanced-level mathematics students in learning exponential equations?
b. How can advanced-level students' understanding of exponential equations be described using APOS theory?

It is hoped that the identification of difficulties experienced by advanced-level students and the genetic decomposition will inform mathematics teachers of some of the conceptual understanding challenges experienced by students in the topic of exponential equations.

LITERATURE REVIEW

According to NCTM (2010), exponential functions are considered to be an important domain in secondary mathematics, although it is seen as one of the challenging topics. Lancelloti (2013) and Sawalha (2018) explained that exponential equations are important in everyday life as they can be used to model

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real-life scenarios, such as understanding growth and decay situations, population growth, and saving accounts. Despite its importance, Jupri et al. (2021) and Tseng (2012) asserted that students tend to use procedural strategies when manipulating exponential equations and lack the aspect of conceptual understanding. Hewson (2013) noticed that students struggle with mastering the laws of exponents and their applications as well as the concepts on logarithms, where they tend to memorize some procedures. This finding coincides with that of Chau (2006), who found that students struggle to manipulate logarithms. Hewson (2013) further outlined that a logarithm function is seen as the inverse of an exponential function and that students can use the method of logarithms when manipulating exponential functions. Jupri and Sispiyati (2020) explained that in order to strengthen students' understanding of learnt concepts, it is important to analyze where the students' weaknesses lie.

Fauzi and Priatna (2022) carried out a study with 30 Grade 8 students to analyze their mathematical communication when solving exponential equations. The study revealed that students' mathematical connections with the concepts are very low and that they need guidance. In another study (Birenbaum & Tatsuoka, 1993), Grade 10 students were able to master the following multiplication and division laws: $a^n.a^m = a^{n+m}$, $\frac{a^n}{a^m} = a^{n-m}$, and $\left(\frac{1}{a}\right)^{n} = \frac{1}{a^n} = a^{-n}$. However, it was observed that students struggled to master these theorems: $a^0 = 1$ and $(a^m)^n = a^{mn}$, and concluded that these two theorems need to be more sufficiently instructed.

Tseng (2012) outlined that when teaching exponential functions, it is important to start with concrete examples before outlining the theorem for conceptual understanding. Another study was carried out by Machisi (2012) on Grade 11 students in the solving of exponential equations of the form $a^{x+p} \pm a^{x+q} = k$. It was noticed that some students struggled to understand equations of this form, while others were able to come up with unexpected methods that did not obey the usual known exponential laws. Based on the study's findings, Machisi (2012) concluded that if one is given the expression of the form $a^x \pm a^{x+p} = k$, where $k = a^y \pm a^{y+p}$, then the following propositions hold: (a) $x = y$, (b) $x + y = y + p$, and (c) $x + x + p = y + y + p$. This shows that a teacher's mathematical knowledge has a strong impact on students' understanding. It is thus important to try and mend this learning gap and determine where students are making mistakes. Conversely, teachers must come up with alternative methods when it comes to exponents and adapt how they teach the topic.

It is against the above background that we used Dubinsky's (1991) APOS theory to understand how students develop their understanding of exponential equations. The APOS levels as suggested by Asiala et al. (1996) are one way of evaluating a student's understanding of a mathematical concept and provide a way for helping the student in that development. For this study, exponential equation will be defined as an equation in which the variables appear as an index. For example, f(x) = 2^x or, more generally, f(x) = a^x. According to Sadler and Thorning (1987), an exponential equation is an equation in which a variable occurs in the exponent. For example, y = 2^x is an exponential equation, since the exponent is the variable x (also said as "2 to the power of x"). This can be generalized to y = ab^x, where a and b are constants and x and y are variables. In addition, a is called the

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initial value and b the base value, and x is considered the independent variable and y the dependent variable.

THEORETICAL FRAMEWORK

In this study, we want to find out how students at the advanced level acquire an understanding of exponential functions, based on a constructivist paradigm. The theory which we chose is APOS theory, which is based on Piaget's theory of abstraction, which has its roots in constructivism (Arnon et al., 2014). It depends on a hypothesis which begins with a mathematical activity developing with students operating at action level. The actions are further interiorized into processes and then encapsulated into objects. Each of the APOS components will be looked at in detail:

a. **Action**: Arnon et al. (2014) asserted that at action level, transformation of objects is externally driven and relies on memorization as well as step-by-step instructions.
b. **Process:** According to Arnon et al. (2014), a process is achieved when actions are repeated by an individual irrespective of relying on external cues. The individual reflects on the procedure and has the ability to image carrying out the steps in the mind or skip some of the steps.
c. **Object:** This is the third level of APOS theory. Dubinsky and MacDonald (2001) explained that an object is derived from the process when the learner is aware of the process in totality. Arnon et al. (2014) further stated that at this level, the individual is able to reflect on the many different representations of the concept.
d. **Schema**: According to Asiala et al. (1996), the schema level is reached when a student's mathematical understanding is at the object level, after which the actions, processes and objects are associated with a specific mathematical concept.
e. In order to develop the concept on exponential equations, we formulated a genetic decomposition, as shown in Figure 1.

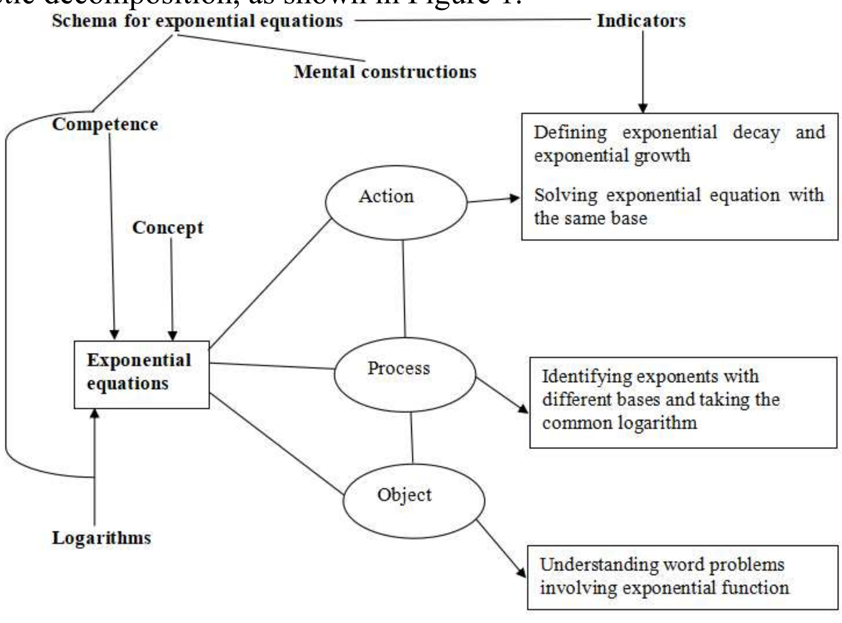

Figure 1. Genetic decomposition of exponential functions

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The implementation of APOS theory as a framework for teaching and learning mathematics involves a theoretical analysis of the concepts under study, called genetic decomposition (Asiala et al., 1996). A genetic decomposition can be explained as a model that describes the mental structures and mechanisms that a student might need to construct in order to learn a specific mathematical concept (Arnon et al., 2014). A genetic decomposition consists of a description of the action that a student needs to perform for an existing mental object. It continues to include explanations of how these actions are interiorized into processes, which are then encapsulated into mental objects so as to be conceived as an entity. Arnon et al. (2014) indicated that a genetic decomposition is a process a researcher follows in an attempt to make sense of how students learn a particular mathematical concept. The genetic decomposition can be useful and revealing, explaining reasons behind the student difficulties. Weyer (2010) stated that genetic decomposition describes the specific mental construction that a learner might make in order to develop their understanding of a given concept. The genetic decomposition of a concept is thus a set of mental constructions which helps to describe how an individual can develop a new learnt concept.

RESEARCH METHOD

Research Design

For this study, we used the interpretive research design. The study was qualitative in nature and a single case study was utilized. This design was chosen because it deals with real-life contexts. Creswell (2014) explained that a case study involves an in-depth exploration of a case, so as to provide a comprehensive account of experiences or events that take place at a particular instance. Creswell (2014) saw qualitative research as a methodical and objective way to describing and giving meaning to life experiences. By its nature, qualitative research allows a researcher to use different research approaches to collect data, such as interviews, document analysis, observations and participation at the research site (Marshall & Rossmann, 2011). Semi-structured interviews and document analysis were used to collect data. A detailed understanding of the concepts under study was developed by identifying themes and patterns emerging from the data.

Participants

The research participants comprised a group of 31 advanced-level students studying at a selected high school in Zimbabwe. These students were doing the Pure Mathematics Syllabus (6042), which includes the topic exponential equations, and the examining board was ZIMSEC. Purposive sampling was used to select the participants, who were doing these concepts at school and thus possessed the particular characteristics required for the study. Purposive sampling allows for the selection of information-rich cases (Kombo & Tromp, 2006).

Data Collection

Three questions on exponential equations were administered to the 31 participants, with data generated through the written responses of the participants. The questions were given to participants after they had been taught by their teacher for seven consecutive school days. Participants were told to show all the working

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and the steps they took to get to all their answers during the writing session. We believe that asking students for explanations will broaden their thinking capacity so that they will be able to solve some challenging life problems. One of the researcher marked the work and conducted follow-up interviews with seven participants based on their written responses. The interviews were semi-structured, which allows probing of the participants to clarify some of their responses and allowing them also to ask for clarification of questions where necessary. Pseudonyms with tags S1 to S31 were used to ascertain confidentiality and anonymity.

Data Analysis

An in-depth content analysis was carried out and was mainly based on the preliminary genetic decomposition. The participants' areas of difficulties, for example failure to find the common base, were noted so as to answer the research questions. Data analysis was also accompanied by participants' written responses to generate rich data. The tasks are represented below.

RESULTS AND DISCUSSION

Participants were given three questions. The first questions tested the action understanding of the concept on exponential equations according to the genetic decomposition in Figure 1. The second question tested the action- and process-level understanding of the concept on exponential equations. Lastly, question 3 tested the object understanding of the concept on exponential equations.

Question 1

Question 1 is aimed at exploring the participant's knowledge of using the method of finding a common base when solving exponential functions and find out whether the participants have developed the action conception of the concept. Question 1 is represented below and a summary of the responses shown in Table 1.

Solve the exponential equation $8^{4x-2} = 32^{-1}$ *without using logarithms*

Table 1. Summary of allocation of scores for question 1

Category	1	2	3	4
Indicator	No attempt, or totally incorrect & left blank spaces, and used logarithms wrongly	Attempted to apply exponent rules but applied them wrongly, or chose the wrong base	Applied the laws of exponents with correct base but encountered errors	Applied the rules of exponents correctly and showed correct answers
Number of participants	6	10	8	7

Seven participants (category 4) gave the correct responses to question 1. They provided a complete and correct indication that they had constructed the suitable mental constructions necessary for developing a conceptual understanding of the concept. They successfully made the correct links on exponent rules and therefore were able to perform the required operation accurately. They were able to choose the correct same base, that is base 2, and raise it to the correct powers. They

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were also able to coordinate the concept of equating the indices and solving the resulting equation correctly. These seven participants proved that they can work with problems in different forms.

Eight participants (category 3) displayed mathematical inaccuracy. Their inaccuracies arose mostly from failure to carry out correct manipulation. These participants made procedural errors, indicating a lack of algorithm skills. Figure 2 presents the response of Participant S3.

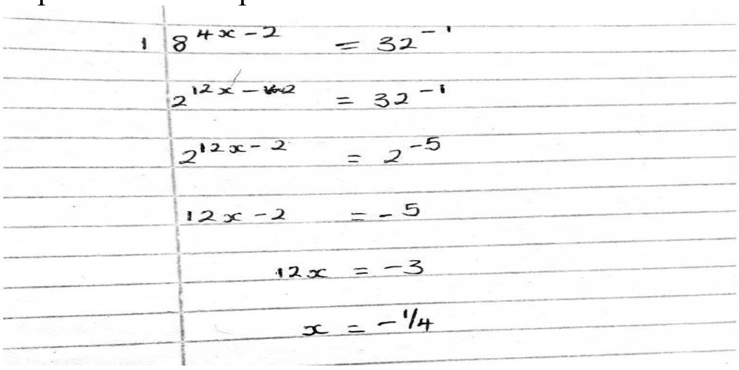

Figure 2. Written response of Participant S3 for question 1

The participant here exhibited confusion on the expansion of brackets. The participant obtained the correct exponent, but instead of multiplying it by $(4x-2)$, he only multiplied it by 4x. Though this may seem to me a minor error, it signals a deeper confusion on the concept of simplification of algebraic expressions. An interview with the participant confirmed that he was still holding on to that error, which hampered him to develop his understanding at the action level according to 993APOS theory.

The other participants made a lot of calculation errors, with failure to solve the resulting linear equations. For example, S29 had the following: $2^{12x-6} = 2^{-5}$, $12x-6=-5=x=-\frac{11}{12}$. This shows that the participant did not grasp the concepts of directed numbers and solving of linear equations. The errors indicated here were as a result of failure to use the equal sign correctly as well as failure to simplify $-5+6$. Quite a number of participants had problems with the simplification of directed numbers. Another participant, S10, was able to find the correct exponents, but came up with the equation of the form $2ln(12x-6) = 2ln(-5)$. This participant showed poor conceptualization of the concepts on exponential equations which resulted in a mixing up of wrong ideas. When the participant was interviewed, he explained that this is how they did it in class, that you need to multiply the exponent by the base and the logarithm. This results in students simply following the rules on simplification of exponential equations without understanding its meaning.

Ten participants (category 2) had an idea that they needed to find the same base, but performed the technique poorly, as illustrated by Participants S5's response in Figure 3.

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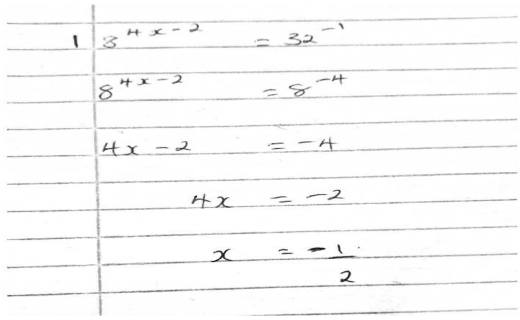

Figure 3. Written response of Participant S5 for question 1

The participant here chose the wrong base. When interviewed, the participant explained that the base 8 is given on the left-hand side so the right-hand side must also have a base 8. He further argued that 8×4 equals 32. When asked to expand 8^{-4}, he was not able to do so. It seems the main confusion was with the negative sign. The researcher asked him to expand 8^3 and he was able to do so. This also shows that the participant did not quite understand the concept of a negative exponent. Nonetheless, another participant in this category, S7, was aware of the theorem involving the negative exponent, as shown in Figure 4.

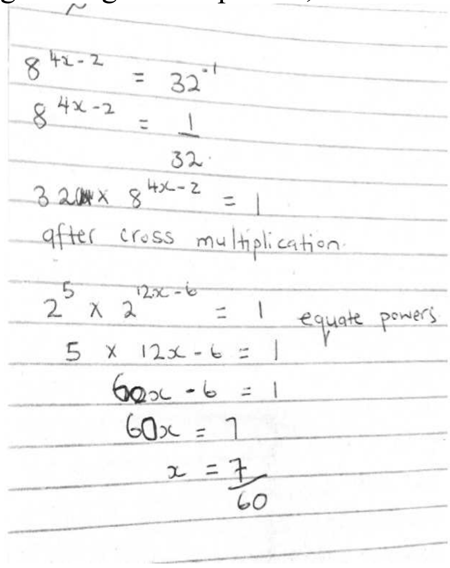

Figure 4. Written response of Participant S7 for question 1

The written response of Participant S7 shows that his solving of the exponential equations was not fully cognitively organized, though he showed some conceptual understanding of some of the concepts. The participant first changed the expression to a positive one, and then proceeded to do cross-multiplication. In his mind, he had an idea about having the same base, but failed to put the same base on the right-hand side. Below is an interview excerpt from the interview with Participant S7:

Researcher	:	*Looking at your solution to question number 1, why did you express* 32^{-1} *to a fraction?*
S7		*I'm applying a certain theorem. It is easier to work with positive numbers. I then removed the denominator by doing cross-multiplication. Then when solving exponential equations, we need same base.*
Researcher	:	*Which same base are you talking about?*

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S7		*We must have the same base 2.*
Researcher	:	*But on the right-hand side there is no base 2.*
S7		*The exponent there is 1. Uhm… we cannot get a base 2.*
Researcher	:	*On the left-hand side, why did you multiply the indices?*
S7		*Not sure now, sir, but we have multiplied the two, so we also multiply it.*
Researcher	:	*What is* 4^0 *or* 10^0*?*
S7		*[Laughing] Any number raised to the power zero the answer is 1.*
Researcher	:	*So, how could you get a base 2 on the right-hand side?*
S7		*Uhhh, not sure.*

From the dialogue with Participant S7, it is evident that the participant realized his misconceptions, and this correlates with Birenbaum and Tatsuoka's (1993) findings. The participant's response shows that he could not abstractly construct the concept of laws of indices. He knew that if you have the same base, you must bring down the powers, so instead of adding them, he proceeded to multiply them. This shows that the theorem $a^b \times a^c$ was poorly conceptualized as well as a^0. The participant could not multiply this theorem of negative power when confronted by a new situation. He was aware that for a^0 the answer 1, but not the reserve part. Five participants could not figure out that 1 is equivalent to 2^0. In terms of APOS, these participants had not fully developed their understanding at the action level according to the genetic decomposition.

Six of the thirty-one participants (category 1) who attempted to answer the question totally missed the point, with some leaving blank spaces. This shows that they were operating at a pre-functional stage, as propounded by Dubinsky (1997). These six participants did not even fit into the action level according to APOS theory. The ability to follow instructions without reasons is an indicator of students operating at the action level. According to the APOS framework, these participants were thus operating below the action level. This means that they learnt the concepts as isolated facts, unable to see the interrelationships between concepts.

Question 2

Question 2 was aimed at exploring participants' knowledge of using the logarithmic method when solving exponential functions. The intention to gain insight into whether or not the participants had developed a process conception of the concept exponential equations and its relationship to other concepts such as use of logarithms as well as logarithmic theorems. The question is presented below and a summary of the allocation of scores is shown in Table 2.

Choose a suitable method to solve the following exponential equations:

$2(18)^x = 6^{x+1}$

$\log (2(18)^x) = \log 6^{x+1}$

$\log 2 + x \log 18 = (x+1) \log 6$

$x \log 18 - x \log 6 = \log 6 - \log 2$

$x (\log 18 - \log 6) = \log 6 - \log 2$

$$x = \frac{\log 6 - \log 2}{\log 18 - \log 6} = 1$$

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Table 2. Summary of allocation of scores for question 2

Category	1	2	3	4
Indicator	No attempt or totally incorrect	Attempted to apply exponent rules and finding the same base	Applied the laws of logarithms but encountered errors	Applied the rules of logarithms correctly and showed correct answers
Number of participants	5	13	8	5

Five participants (category 4) provided complete and correct responses to this question. This is an indication that they had constructed the necessary mental construction for developing a conceptual understanding of the concept of exponential equations using laws of logarithms according to the genetic decomposition. These participants were able to rewrite the exponential equation in logarithmic form, and applied the addition and subtraction laws of logarithms. Based on the responses, it seems that these participants had interiorized the action into the process understanding of solving exponential equations. A discussion with Participant S10 showed that he had conceptualized the concept and really has an understanding at the process level. According to Dubinsky and Macdonald (2001), in his study also noted that students actually work with problems in different forms.

Eight of the participants (category 3) were aware that they were solving exponential equations with different bases to the extent that they used the concept that the inverse of an exponential function is a logarithm. However, these participants experienced a number of shortcomings, as shown by Participant S23's response (Figure 5).

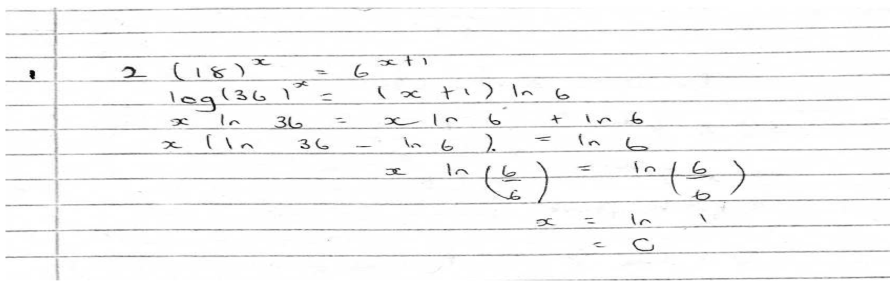

Figure 5. Written response of Participant S23 for question 2

Participants S23's response shows that the participant knew how to apply the laws of logarithms but struggled to apply them. The participant multiplied 2 by an exponential function, which is incorrect. Correct procedures are seen, however, as the participant was able to apply the rules of logarithms on both sides. We also noticed that on the right-hand side, the participant was able to distribute ln 6 inside the parenthesis, though the brackets are missing. The participant was able to correct like terms together and correct factorization was evident. However, the participant lacked the technique of division of logarithms and displayed a mathematical inaccuracy, and this showed that this participant lacked the background knowledge of the laws of logarithms. He could not perform the required operation correctly.

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He failed to distinguish between $ln\left(\frac{6}{6}\right)$ and $\left(\frac{ln\,6}{ln\,6}\right)$. This hindered the participant from developing his understanding at the process level. The concept on division of logarithms and multiplication by an exponent has to be included in the modified genetic decomposition. Some participants failed to apply the logarithm rules correctly, preventing them from developing their understanding at the process level. During the interview with Participant S23, the following discussion took place:

Researcher	:	*In question 2, from your written response, why is it that you multiplied 2 and* 18^x*?*
S23	:	*I wanted to apply the laws of logarithms so I should have only one expression on the left and another expression on the right-hand side. Also, when multiplying algebraic expressions, you do not put into consideration the idea of like terms, thus why I combined the two.*
Researcher	:	*Oh ok, why did you divide* $\ln 6$ *by 6 on the left-hand side? What rules are you using?*
S23	:	*Law of logarithms.*
Researcher	:	*Can you briefly outline the law that you used?*
S23	:	*Hmmm, I have forgotten, sir.*

According to the dialogue between the researcher and Participant S23, the image in the participant's mind was not about exponential properties but in solving the problem. This indicates that many students have the tendency of applying rules even if they do not understand them, since the rule of multiplying a whole number by an exponential expression was never discussed during lesson delivery. What was evident here is that the participant failed to interpret the nature of the problem. Thus, in terms of APOS levels, the action conception of the concept had not fully developed. In the genetic decomposition, the participant had not developed an understanding of the structure of exponential equations. As a result, he applied the wrong procedures. According to Siyepu (2013), such errors persist due to sacrifice-level procedures, where an individual acquires knowledge by heart without engaging with its meaning.

It is interesting to notice that a considerable number of participants used the method of rewriting the exponential equation using the same base in order to solve the system of exponential equations. This was illustrated by 13 of the participants (category 2). Examples can be seen in the responses of Participants S9 and S3 (Figure 6).

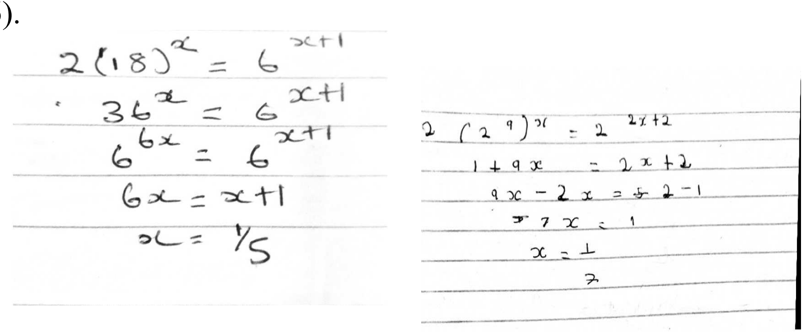

Figure 6. Written responses of Participants S9 and S3, respectively, for question 2

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From the written responses above can be seen that the two participants attempted to express the exponential equations using the same base. These participants simply carried out procedures without constructing the meaning of the concept of laws of indices. The aspect of the third law, for example $a^3 = a \times a \times a$, had not been cognitively constructed. It also seems as though these participants simply used guess and checking strategies. When they were asked about the exponential properties of multiplication and division, neither participant could explicitly explain the difference between multiplication and division of indices.

Their responses indicated that the schema of basic laws of indices had not been developed and it impacted negatively on the new knowledge learnt. We decided to include this aspect in the modified genetic decomposition. This really has a huge implication for students' understanding of the concept on exponential equations, so it needs to be rectified before it leads to future learning barriers in the understanding of exponential equations.

We noticed that all five participants in category 4 attempted the question but they totally missed the ideas. This shows that they were operating at the pre-functional stage, as propounded by Dubinsky and Harel (1992). These five participants did not even fit into the action level according to APOS theory.

Question 3

Question 3 tested the participants' understanding of exponential equations to word problems. The participants were asked to convert the word problem into an exponential equation. The question is presented below and a summary of the allocation of scores is shown in Table 3.

- ***(a)*** *Explain the difference between exponential growth and exponential decay.*
- ***(b)*** *Carbon 14 has a half-life of 5750 years. If initially there are 60 grams of carbon 14, how many grams are left after 3000 years, explaining whether this is exponential growth or exponential decay.*

Table 3. Summary of allocation of scores for question 3

Category	**1**	**2**	**3**
Indicator	No attempt or totally incorrect	Correct response but wrong explanations	Totally correct response with explanations
Number of participants	18	7	6

Most of the participants encountered difficulties in converting word problems into algebraic equations, with 18 out of 31 not understanding the question. They attempted the question but failed to produce the correct answers. Participants confused the parameters that were given to the extent that various meaningless expressions were seen. The word half-life seems to have confused the participants. This shows that they were at the pre-action level according to APOS theory. Out of the 18 participants, some left blank spaces. In an attempt to distinguish the terms exponential growth and exponential decay, the following explanations were evident in some of the students written work:

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a. Growth exponentially means as 'x' value increases 'y' also increases; they are directly proportional.
b. Decay exponentially means an increase in 'x' results in a decrease in 'y'; they are inversely proportional.
c. To grow and decay exponentially means to increase and decrease.
d. To grow exponentially means to rapidly grow.
e. To decay exponentially means to decrease very fast.

The above explanations show that the participants struggled to define these terms. These participants did not develop any action conceptions of the difference between the two terms. Some first divided 5760 by half, and various expressions were evident. The extract below (Figure 7) shows the written solution by Participant S12, who did not answer part a and struggled to answer part b of the question.

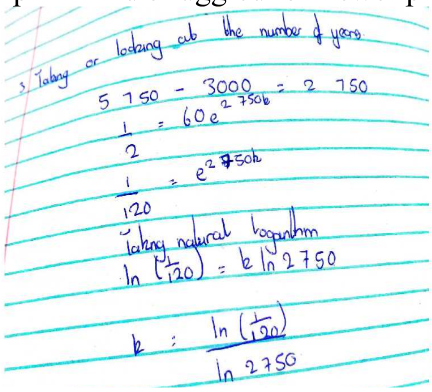

Figure 7. Written response of Participant S12 for question 3

Participant S12 had an idea about the initial conditions but struggled to produce the correct expression. The participant started by subtracting 3000 from 5760. This stage was incorrect. However, the participant had the correct initial conditions but was confused with the term half-life. Instead of putting $\frac{1}{2}$ of 60 the participant simply put $\frac{1}{2}$. The participant also did not justify his result, whether it is exponential decay or exponential growth. He was asked open-ended questions in order to probe his conceptual understanding. The following dialogue took place between the one of the researcher and the participant:

Researcher : *Can you explain the difference between exponential growth and exponential decay? I can see you did not attempt this question.*
S12 : *Exponential growth means increasing an increase in the y coordinate, whereas to decay is decrease in the y coordinate.*
Researcher : *Oh ok. It's increasing and reduction in terms of what or in relation to what?*
S12 : *Not sure, but we calculate it using the decay formula.*
Researcher : *Can you define the term half-life of a radioactive decay?*
S12 : *Umm, I don't still remember.*
Researcher : *Why did you subtract 3000 from 5760?*
S12 : *I could not exactly figure out where I was supposed to use the two, so I had to subtract.*
Researcher : *So, why didn't you outline whether it is exponential decay or not?*
S12 : *I was stuck because after finding the value k I did not know what to do next.*

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The extract shows that this participant faced difficulties in solving word problems involving exponential equations. He could not even make sense of the questions asked. This was also witnessed by Jones (2006), who outlined that students have trouble with the language of functions.

Seven of the thirty-one participants (category 2) constructed the collection of rules and assimilated them in their cognitive structures but did not interiorize them into a process as they failed to explain the concepts of exponential growth and exponential decay. They also could not outline whether the resulting solution was an exponential decay or not. This hindered these participants from developing their understanding at the object level according to APOS theory.

Six of the thirty-one participants provided a complete and correct indication that they had constructed the suitable mental constructions necessary for developing a conceptual understanding of the concept. These participants proved that they can actually work with problems in different forms and could encapsulate the process into a cognitive object (Dubinsky& McDonald, 2001).

The purpose of this study was to determine the kinds and sources of difficulties encountered by students when solving exponential equations as well as exploring their mental constructions when learning these concepts. Analysis of the written responses and interviews revealed that most participants had memorized the algorithms without conceptualizing the learnt material. We noticed most of the participants applied the rule of exponents wrongly as they struggled to choose a common base. This is because if the base is not the same, critical reasoning is needed to think what number raised to a power will be equal to the original base. Thus, both bases must be changed to the same base. For example, participants failed to have a common base 2 by failing to express 1 as 2^0. Birenbaum and Tatsuoka (1993) also noticed that students struggle to understand the idea that where any number is raised to the power 0 the answer is 1. Some of the participants lacked some algorithmic skills of basic algebra, as they failed to expand brackets. Others struggled with the number system by failing to add or subtract directed numbers. This is in line with Kazunga and Bansilal's (2020) finding that the difficulties that students have with prior knowledge prevent them from working more efficiently with higher level concepts.

For question 2, most of the participants failed to apply the logarithm rules. In their minds, they always think of finding a common base. Some of the participants knew that they needed to use the logarithm rules, but in the process failed to apply the correct rules. Participants struggled to use the following laws of logarithms: $log\ M - log\ N = log\left(\frac{M}{N}\right)$, $log\ M + log\ N = log\ (MN)$ and $log\ M \div log\ N = \frac{log\ M}{log\ N}$. The language of mathematics also became a barrier for them to make sense of the questions when they were asked to convert word problems involving exponential equation to algebraic statements and were not able to translate the statements into symbols. This supports the finding of Noutsara et al. (2021) with regard to the difficulties that students encounter when dealing with word problems. These authors argued that many students encounter a lack of understanding when working with number stories as they lack proficiency to the extent that they experience reading as well as comprehension errors. Most of the

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errors demonstrated by these participants were mainly computational and conceptual in nature as well as mixing up of rules. Students must possess both procedural and conceptual knowledge so that they are in a position to solve challenging problems. Related literature has revealed that understanding the notions of logarithm and exponent rules is fundamental to understanding exponential equations. Understanding students' thinking is very crucial when they are solving mathematical problems, and there is a need to be more tolerant with students who struggle to understand concepts (Hunt & Little, 2019). It is recommended that prior to learning exponential equations, students should understand the notions of law of indices, logarithms, expansion of brackets and simplification of directed numbers.

CONCLUSION

The APOS analysis in this study was guided by the preliminary genetic decomposition explained under theoretical framework. It has also been revealed that students build their understanding of exponential equations at different levels as defined by APOS theory. The written responses of participants revealed that in terms of APOS, 15 of the participants (49%) had developed at least an action conception of solving exponential equations with the common base. Eight participants (26%) struggled to find a common base, indicating that the action conception was not fully developed. The remaining six (19%) were operating at the pre-functional level as they did not follow the given instructions and used a wrong method to answer the given problem.

The findings also revealed that only five of the participants (16%) had developed their understanding at the process level of solving exponential equations with different bases. Some of the participants encountered so many errors that this hindered them to develop their understanding at the process level according to APOS theory. Eight (26%) had developed their understanding at the action level according to APOS theory as they encountered some theoretical difficulties in the process, which hampered them from developing their understanding at the process level according to APOS theory. This shows that they had learnt these concepts by heart. The remaining 18 participants (58%) applied the wrong method and were thus operating below the action level.

It is also evident from this research that participants had limited understanding of exponential equations as they performed poorly on the concept of radioactive decay, which was not well grasped. According to the APOS levels, most participants were operating below the process level. Very few participants, that is 5 (16%) were operating at the object level according to APOS theory. This supports the contention by De Lima and Tall (2008) that it is very challenging for students to move from the process level of understanding to the object level of understanding.

The abstract nature of exponential equations was identified as the chief source of errors. Using the APOS levels, most participants showed that they were operating at the action and pre-functional level.

RECOMMENDATIONS

Teachers are encouraged to embrace the difficulties and misconceptions made by learners when solving exponential equations. The difficulties students have with the solving of exponential equations is caused by lack of background

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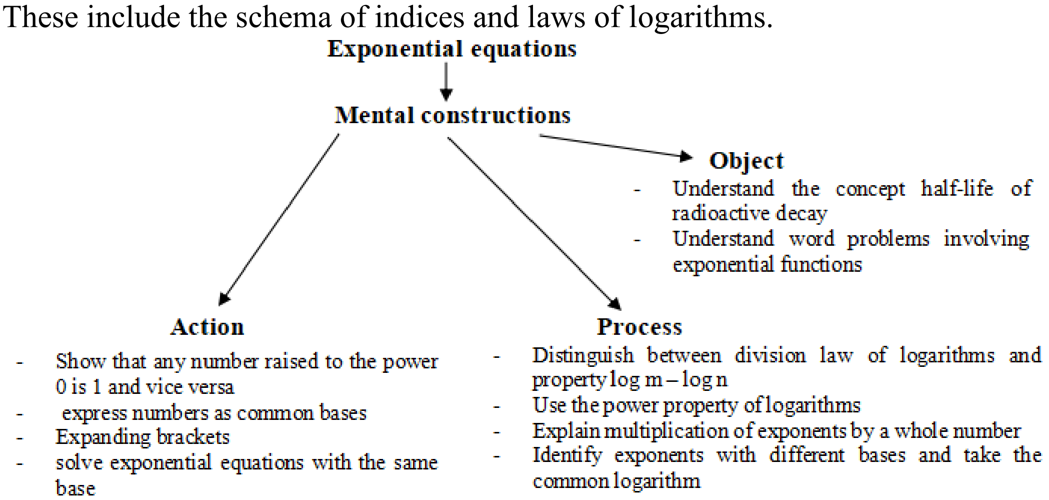
concepts. Mason (2015) outlined that if concepts are always introduced through examples, or through definitions followed by examples, students are more likely to conceptualize the concepts. Teachers are encouraged to put more emphasis on topics such as the law of indices and law of logarithms at the ordinary level so that students can understand exponential equations at the advanced level. The link to other concepts must thus be highlighted to reduce errors (Msomi & Bansilal, 2022). Furthermore, students should be given more opportunities and more structured examples on radioactive decay. It is important to explain the definition of a concept through examples. Instructors can also make use of error analysis as it aids students to discover their own errors (Rushton, 2018). We strongly recommend that teachers organize school-based or cluster-based workshops so as to extend expertise and deepen mathematics pedagogical content and to strategize how to clarify exponential equation mythology.

As part of the pedagogical considerations, we provide a genetic decomposition for exponential equation concepts (Figure 8 below). We hope that this will result in instructional treatment that would guide students to make the necessary mental constructions relevant to exponential functions and lead to improvement of their understanding of relevant concepts.

STUDY LIMITATIONS AND SUGGESTIONS FOR FURTHER EXPLORATION

This study was done during the Covid-19 pandemic, where movement of both teachers and learners was restricted. This led us to concentrate on form 6 mathematics students at one high school only. Nonetheless, we hope that the genetic decomposition presented is rich enough to be adopted by other teachers teaching the same concept at other schools in Zimbabwe. However, our reflections on the teaching design indicate that more time needs to be devoted to helping students develop the mental structures at the process, object and schema levels.

We noticed that although the genetic decomposition was useful as our diagnostic tool, some concepts are essential for the conceptual development of exponential equations. Hence, we suggest that it be modified to accommodate some useful concepts needed for conceptual development of exponential equations. These include the schema of indices and laws of logarithms.



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