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Mathematics Education Undergraduates' Conceptualisation of Non-Homogeneous Differential Equations Using the Method of Undetermined Coefficients

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Abstract

The topic of ordinary differential equations (ODEs) is widely covered in higher education institutions, however, understanding non-homogeneous differential equations (NHDEs) poses challenges for students due to its complexity. Limited research in second order NHDEs points to the need for more studies on students' conceptions and understanding of this topic and its basic concepts. This study delved into the conceptualisation of second order NHDEs and their solutions using the method of undetermined coefficients among Bachelor of Science in Education Mathematics student teachers. Employing the APOS Theory, an in-depth test on NHDE was administered to 60 participants from a university in Zimbabwe to identify their underlying challenges. The study revealed that student teachers encountered difficulties in determining general solutions for NHDEs using the method of undetermined coefficients, largely due to insufficient knowledge of basic ODEs concepts such as differentiation of exponentials, trigonometric ratios, solving quadratic equations, and algebraic skills. The findings indicated that the participants primarily utilised reasoning associated with APOS 'action' and less of 'process', hindering their engagement with higher-level concepts of solving NHDEs. Consequently, enhancing student teachers' performance and retention in NHDEs classes therefore requires inventive approaches to instruction.

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INTRODUCTION

Non-homogeneous differential equations (NHDEs) represent a distinct category within ordinary differential equations (ODEs). ODEs hold a fundamental position in the field of mathematics and have been an integral component of calculus for centuries (Aisha et al., 2017). Their conceptual framework serves to model and comprehend real-world issues, finding application in various physical phenomena across the natural sciences (Boyce & Diprima, 2012; Zill, 2018). Moreover, ODEs have proven instrumental in addressing everyday challenges such as bacterial growth rates and heating and cooling processes (Farlina et al., 2018). The significance of ODEs is reflected in their inclusion as essential subjects across diverse departments in higher education (Arslan, 2010). Tsoularis (2021) reinforces

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this view, highlighting the pervasive utility of ODEs in higher education, particularly across physical sciences, business, medicine, biology, and other STEM disciplines. Additionally, ODEs play a pivotal role in comprehending complex engineering concepts. Consequently, ODEs have become integral to the core and prerequisite curriculum of numerous undergraduate programs in Zimbabwean higher education institutions. Their indispensability is further underscored by their inclusion in mathematics education courses across the country's universities, forming a core component of the BScEd Mathematics program.

However, despite their applications in many fields, the conceptualization of differential equations, particularly second order non-homogeneous differential equations, presents challenges for students across various fields. Studies by Luneta and Makonye (2010), and Maat and Zakaria (2011) have highlighted the prevalent difficulties faced by schools in addressing ordinary differential equations (ODEs). Additionally, Farlina et al. (2018) have noted persistent challenges among students in solving ODEs, particularly related to differentiation and integration, which serve as foundational concepts for first and second order differential equations. These challenges often stem from errors and misconceptions in fundamental topics such as the power rule, chain rule, and exponentials (Chikwanha, 2021). Furthermore, students' proficiency in ODEs remains relatively low, leading to difficulties in solving real-world applied questions (Yarman et al., 2020). Jojo (2011) also observed difficulties among students in comprehending and applying calculus concepts, particularly the chain rule.

Arslan (2010) highlights the complexity of concepts related to Ordinary Differential Equations (ODEs), suggesting that students may encounter challenges in comprehending the subject matter. These challenges can impact the performance of in-service teachers in Non-Homogeneous Differential Equations (NHDEs) due to the interconnected nature of the concepts. Yarman et al. (2020) emphasize the significance of ODEs within the context of an applicable curriculum. Therefore, it is imperative to assess the knowledge of in-service teachers regarding NHDEs to enhance their understanding of calculus at the high school level, given its calculus-based nature.

The difficulties associated with solving second-order non-homogeneous differential equations (NHDEs) often stem from students' lack of both conceptual and procedural understanding, resulting in consistently poor performance (Prawoto et al., 2018). These challenges manifest in the form of errors made by students when tackling NHDE problems. Students frequently struggle to grasp solution methods, including the method of undetermined coefficients. Scholarly literature has documented instances where students arrived at correct conclusions in NHDEs, masking underlying misconceptions and conceptual gaps (Rasmussen & Whitehead, 2003). Ningsih and Rohana (2018) assert that to address this issue, students must have a firm grasp of fundamental concepts such as the complementary function and the particular integral. Consequently, those lacking an understanding of these concepts are likely to encounter difficulties in determining solutions for second-order NHDEs.

The significance of ordinary differential equations (ODEs) has garnered considerable attention from researchers, primarily focusing on ODEs' content and instruction (Arslan, 2010). However, the scarcity of research on second-order nonhomogeneous differential equations (NHDEs) emphasizes the necessity for further studies on students' conceptions and comprehension of NHDEs and their fundamental principles. Examining learners' performance in NHDEs can offer valuable insights for tailoring instruction to address learners' specific needs in this area. It is crucial to analyze and rectify errors at this level to enhance the teaching of calculus in high school, as NHDEs are a calculus driven topic (Luneta & Makonye, 2010). This study aims to unravel the challenges and opportunities experienced by BScEd mathematics learners in response to NHDEs test questions. Its purpose is to explore BScEd mathematics students' conceptualization of second-order NHDEs and their solutions using the undetermined coefficients method.

The anatomy of ODES

An ODE is an equation containing one or more derivatives (Ningsih & Mulbasan, 2009). ODEs can be named after their order. For example, a second order differential equation is an ODE where the order of the highest derivative is two. An ODE in the form:

$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$ is a second order differential equation ….(1)

When writing an ODE, there are three types of derivative notations that can be used.

a. Lagrange's notation.

This notation expresses the derivative of the function f as f', hence, a second order ODE in (1) is written in the form $y'' + p(x)y' + q(x)y = f(x)$ …………..(2)

b. The Leibniz's notation

The derivative of f is expressed as in (1) above. That is $\frac{d}{dx}f(x)$, and if $y = f(x)$, then the derivative is $\frac{dy}{dx}$.

c. Newton's notation (Dot notation)

The derivative of the function f in (1) is denoted by $a_2\ddot{y} + a_1\dot{y} + a_0y = f(x)$ ……..(3)

In this study, the Lagrange's notation will be commonly used.

Homogeneous/non-Homogeneous second order ODEs

An ordinary differential equation (ODE) is an equation that only contains ordinary derivatives of one or more functions with respect to a single independent variable (Zill, 2013). ODEs can be classified into two types based on their properties: homogeneous and non-homogeneous linear differential equations. A second-order homogeneous equation is represented in the form $ay'' + by' + cy = 0$. In this form, there are no other terms that are either constants or only a function of x. If such terms were present, it would be conventional to collect them on the right-hand side of the equation. However, because they are not present, a zero is placed on the right-hand side of the equation. This type of ODE is termed as homogeneous (McDonald, 2004). Alternatively, when the equation includes terms that are functions of x only or constants, it can be expressed in the form $ay'' + by' + cy = f(x)$, where the function f(x) is not equal to zero, and a, b, c are constants. This type of ODE is termed as non-homogeneous (McDonald, 2004; Boyce & DiPrima, 2012). There are various methods for solving non-homogeneous differential equations, including variation of parameters, power series solutions, the Laplace

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transform, and undetermined coefficients. This study focuses on undetermined coefficients.

Solution of Homogeneous Equations

Taking heed of Zill's (2013) method of solving a homogeneous equation, the first attempt is a solution for the form e^{mx} . This produces a characteristic equation to be solved for m. The general roots of the homogeneous equation are obtained from the roots of the characteristic equation. For example: solve y'' + 4y' + 3y = 0.

To find the characteristic equation: (i) Let the solution y(x) = e^{mx}, then y^1(x) = m e^{mx}, y'' = m^2e^{mx}.

Substituting in the ODE, we have: m^2e^{mx} +4 m e^{mx} + 3 e^{mx} = 0.

Dividing throughout by e^{mx}, we remain with m^2 +4 m + 3 = 0 (characteristic equation). This is a quadratic equation, so it has two roots m_1 and m_2 with three possible forms:

(i) Real distinct roots: $m_1 \neq m_2$
(ii) Repeated roots: m_1 = m_2 = m
(iii) Complex roots: $m_{1,2}$ = a ± bi

The general solution: y_c = y_1(x) + y_2(x), where y_1(x) and y_2(x) are the two roots determined by the values of m ………………………………………… (A)

Therefore, for the roots in (i) (real distinct roots), the general solution y_c of the H/E then becomes:

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$. ………………………………………(B)

For the roots in (iii), y_c = e^{a+bi} = e^{ax}(cos bx + i sin bx) + e^{a-bi} = e^{ax}(cos bx – i sin bx)

y_c = e^{ax} (cos bx + i sin bx) + e^{ax} (cos bx – i sin bx)

y_c = 2 e^{ax} cos bx after expanding the brackets

For the roots in (ii) since m_1 = m_2, y_1(x) = y_2(x) = e^{mx}

These are the same solutions and will NOT be "good enough" to form a general solution. To find the second solution, we consider the fact that a constant multiplied by a solution for a linear homogeneous ODE is also a solution. Hence y_1(x) = e^{mx} , whilst y_2(x) can be written as c_2x e^{mx}. Therefore, the general solution y_c = c_1 e^{mx} + c_2x e^{mx} for repeated roots.

Finding Particular Solution Using Undetermined coefficients

Finding solutions for second-order non-homogeneous differential equations (NHDEs) is a challenging task, and explicit solutions are only achievable in specific

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cases. The general solution for second-order non-homogeneous ordinary differential equations (ODEs) comprises the sum of solutions for the associated homogeneous differential equation and the particular solution for the nonhomogeneous equation. According to Arficho (2015), the general solution for NHDEs represents a solution for a differential equation with arbitrary parameters, while a particular solution is a solution for a differential equation that is free of arbitrary parameters.

The method of undetermined coefficients relies on a conjecture, an educated guess about the form of the particular integral, driven by the types of functions present in the input function f(x) from the NHDEs, as expressed by Zill (2018). This approach is generally limited to non-homogeneous linear DEs with constant coefficients, where f(x) is a constant, a polynomial function, exponential function, trigonometric function (sine βx or cosine βx), or a finite sum and/or product of these functions. In essence, the method of undetermined coefficients is an effective strategy for finding a particular solution for NHDEs, involving an initial "guess" of the appropriate form, followed by testing through differentiation of the resulting equation (Zill, 2018).

According to Zill (2018), the method of undetermined coefficients states that, when one finds the solution *y* and plugs it on the LHS of the equation, we will end up with g(x), since g(x) is a function of x. One can also guess the form of the particular solution y_p(x) up to arbitrary coefficients and find those coefficients by plugging y_p(x) into the ODE (Holzner, 2008). For example, g(x) can be in the form of an exponential function e^{rx}. The particular solution guess for this type of g(x) will be Ae^{rx} because derivatives of e^{rx} reproduce e^{rx}. In this way, one has a chance of finding a particular solution in this form. Furthermore, g(x) can also be in the form of a polynomial of order n. For example, g(x) = 9x^2 – 10*x*. In this case, the guess of y_p(x) = Ax^2 + B*x* + C (ignoring the coefficients 9 and 10). Moreover, g(*x*) can be a trig ratio sin*x* or cos*x* or a combination of the two. For example, y'' - 4y' - 12y = sin *2x*. In this case, the initial guess of y_p(x) becomes Acos (*2x*) + Bsin (*2x*). Although the RHS of the ODE only has sin *2x* without cos *2x*, the derivative of sin*x* produces cos*x*, hence y_p(x) = Acos (*2x*) + Bsin (*2x*). Table 1 below summarises a few examples of the initial guesses for different forms of g(*x*).

Table 1: Examples of Initial Guesses

g(x)	y_p**(x) (guess)**
a$e^{\beta x}$	$Ae^{\beta x}$
a$\cos\beta x$	A$\cos\beta x$ + B$\sin\beta x$
b$\sin\beta x$	A$\cos\beta x$ + B$\sin\beta x$
a$\cos\beta x$ + b$\sin\beta x$	A$\cos\beta x$ + B$\sin\beta x$
$(9x^2 - 10x)\cos x$	**$(Ax^2 + Bx + C)\cos x + (Dx^2 + Ex + F)\sin x$
$4x - 5 + 6xe^{2x}$	$Ax + B + C\,xe^{2x} + Ee^{2x}$

Take note that when writing the initial guesses, the constant coefficients are ignored.

For ** in Table 1, the initial guess is found using the following procedure:

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$9x^2 - 10x = Ax^2 + Bx + C$ (coefficients 9 & 10 are ignored); and for cos x = Dcos$x$ + Esin$x$, where A,B,C,D,E are constants.

Multiplying the two gives:

$(Ax^2 + Bx + C)(D\cos x + E\sin x)$
$= (Ax^2 + Bx + C)(D\cos x) + (Ax^2 + Bx + C)(E\sin x)$
$= (ADx^2 + BDx + CD)(\cos x) + (AEx^2 + BEx + CE)(\sin x)$

Since A,B,C,D,E are constants, we can rename the constants to:

$(Ax^2 + Bx + C)\cos x + (Dx^2 + Ex + F)\sin x$

By and large, the overall solution for the 2nd order NHDEs using the method of undetermined coefficient is y = y_c (*complementary solution*) + y_p (*particular* solution) where $y_c = c_1y_1(x) + c_2y_2(x)$, (A) hence, $y = c_1y_1(x) + c_2y_2(x) + y_p(x)$.
In general, the solution for a second order linear NHDEs using the method of undetermined coefficients follows the following procedures as determined by Holzner (2008):

(i) Find the corresponding H/E by setting g(x) = 0. Form and solve the characteristic equation to find y_c , as in (A) above.
(ii) Find the particular solution y_p**(x)** using initial guesses as in Table 1 above.
(iii) The general solution for the non-homogeneous equation is therefore: $y = y_c + y_p$

Example **(**extracted from Holzner, 2008**):**

$y'' - 4y' - 12y = x^2 + 1$ ....(4)

1. The corresponding homogeneous equation is: $y'' - 4y' - 12y = 0$
   Characteristic equation is: $m^2 - 4m - 12 = 0$
   $(m-6)(m+2) = 0$
   Hence m = 6 or −2 (Distinct roots) → $y_c = c_1e^{6x} + c_2e^{-2x}$

2. To find the particular solution, the initial guess of $x^2 + 1$ is $Ax^2 + Bx + C$
   $\therefore y_p(x)$ = <u>$Ax^2 + Bx + C$</u>, $y'_p = 2Ax + B$, $y''_p = 2A$
   Substituting this in (4), $2A - 4(2Ax + B) - 12(Ax^2 + Bx + C) = x^2 + 1$
   $2A - 8A - 4B - 12Ax^2 - 12Bx - 12C = x^2 + 1$
   Equating coefficients, $-12A = 1$, $A = \frac{-1}{12}$
   $-12B = 0$, $B = 0$
   $-8A + 2A - 4B - 12C = 1$
   $-6A - 4B - 12C = 1$
   $-6(\frac{-1}{12}) - 0 - 12C = 1$
   $\frac{1}{2} - 12C = 1$
   $-12C = \frac{1}{2}$
   $C = \frac{-1}{24}$

Therefore $y_p(x)$ = <u>$Ax^2 + Bx + C$</u> $= \frac{-1}{12}x^2 - \frac{1}{24}$
Therefore, the general solution is: $y = y_c + y_p$

$y = c_1e^{6x} + c_2e^{-2x} - \frac{1}{12}x^2 - \frac{1}{24}$

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Challenges in Solving ODEs using the Undetermined Coefficients Method

By and large, in the realm of differential equations, the process of solving an ordinary differential equation (ODE) is acknowledged to be more intricate than mere differentiation, as articulated by Nykamp (2015). Ningsih and Mulbasani (2019) along with Prawoto et al. (2018) emphasize that students often encounter challenges when tackling ODEs due to their tendency to interpret solutions in numerical terms rather than as functions. Furthermore, Ningsih and Rohana (2018) posit that a comprehensive grasp of fundamental concepts such as exponentials, logarithms, derivatives, and integral functions can significantly facilitate the resolution of ODEs. The scholars assert that students frequently grapple with both conceptual and procedural hurdles when dealing with second-order ODEs. This study aimed to scrutinize students' proficiency in solving second-order non-homogeneous differential equations (NHDEs) utilizing the method of undetermined coefficients.

Theoretical Framework

This study is underpinned by the Action, Process, Object, and Schema (APOS) theory, as proposed by Dubinsky and Tall (1991), for investigating students' conceptual comprehension of non-homogeneous differential equations. Dubinsky (1991) defines APOS as a theory of learning mathematics that facilitates the understanding of the process of concept formation in mathematics. The APOS theory comprises four stages: action, process, object, and schema. In this study, the APOS theory provides insights into how learners conceptualise and comprehend mathematical concepts. These four primary stages are referred to as mental structures for constructing mathematical knowledge. The theory asserts that the formation of a mathematical concept necessitates the transformation of one entity to obtain another entity. Therefore, a transformation commences with an action, which is internalised to form a process, and subsequently, the process is encapsulated to form objects (Dubinsky et al., 2005). The mental structures are delineated below:

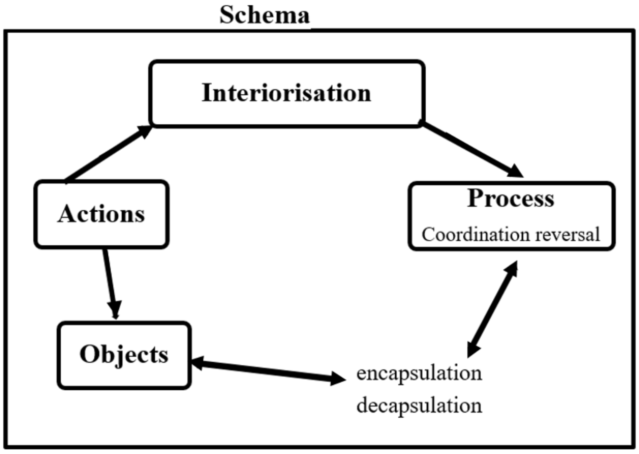

Figure 1. Mental structures for the construction of mathematical concept (Amon et al., 2014, p.10)

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The APOS theory, in the context of constructivist learning in mathematics, emphasises the construction of knowledge through the comprehension of mathematical concepts and the formation of mental objects (Meel, 2003). This theory underscores the importance of paying attention to the process of acquiring new knowledge to support the expected learning outcomes in mathematics. Rooted in Piaget's theory of reflective abstraction, the APOS theory explains how new processes, objects, and schemas are developed to form abstract mathematical concepts (Dubinsky & Tall, 1991). The process of knowledge construction in reflective abstraction involves interiorisation, encapsulation, and decapsulation (reversal), enabling students to navigate back and forth to achieve a comprehensive understanding of a concept. In this specific study, the APOS theory generated testable predictions, suggesting that a learner who constructs a particular set of actions, processes, objects, and schemas in a specific manner is likely to succeed in solving NHDEs of undetermined coefficients. Consequently, the APOS theory was used to make an initial decomposition of what it could mean to understand the solution for NHDEs (Dubnisky & McDonald, 2005) and subsequently analyse the test results. Essentially, the APOS theory was instrumental in categorising students' conceptual comprehension of NHDEs, aiding in the assessment of participants' understanding of the solution for the complementary equation, the formulation of initial "guesses," and the identification of the particular solution for the NHDEs.

RESEARCH METHOD

The researchers found it essential to investigate the conceptualisation of second order nonhomogeneous differential equations (NHDEs) among student teachers. They utilised the method of undetermined coefficients as this is where most pitfalls in ordinary differential equations (ODEs) arise. To gain insights into students' difficulties and achievements, an in-depth NHDEs test was administered to 163 second-year BScEd (mathematics) students at a university in Zimbabwe. These students, who held diplomas in mathematics education from various teachers' training colleges, were qualified to teach mathematics up to ordinary level (form 4) and were actively teaching in schools across Zimbabwe. They were pursuing further education through the Open and Distance e-Learning (ODeL) mode, a block release program that involved attending face-to-face lectures with instructors for four weeks during school holidays. It is important to note that the curriculum of the BScEd program offered at state universities in Zimbabwe is the same.

In this research study, the student teachers were instructed on the subject matter for a duration of one week, following which they underwent a test. The test aimed to assess the student teachers' capacity to exhibit their comprehension and proficiency in solving a range of problems utilising the undetermined coefficients method. The test consisted of 5 items designed to uncover the participants' grasp of concepts, procedural adeptness, adaptive reasoning, strategic competence, and productive disposition in second order non-homogeneous differential equations. Overall, the test aimed to evaluate the student teachers' proficiency in non-homogeneous ordinary differential equations.

All scripts were marked but only 60 participants' results were analysed for the study because most responses were quite repetitive. Moreover, the feedback

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gathered from the 60 students was not meticulously examined on an individual basis. Instead, the researchers opted for a broader approach, grouping the responses into categories based on overarching themes of similarity, overlooking the subtle distinctions that each student might have contributed. Low achievers, intermediate and high achievers were considered to form a group of the 60 participants. The results and findings were therefore reported on these 60 participants only, placed in categories. Discussions with the participants during lectures clarified some misconceptions that they had on the topic. The APOS theory was used to make an initial decomposition of what it could mean to understand the concept of solution for NHDEs and to subsequently analyse the results of the test. The participants' understanding of the parameters was therefore included in the complementary solution. Using the APOS theory, the researchers analysed participants' flexibility in making and manipulating an initial "guess" to the appropriate form and their ability to differentiate and substitute the guesses in the resulting equation to find the numerical values of the undetermined coefficients.

RESULTS AND DISCUSSION

In the assessment, a notable portion of participants opted not to respond to certain questions. This behavior may indicate a tendency to give up when encountering challenges. Unanswered questions could suggest a lack of proficiency in a specific area (Herholdt & Sapire, 2014), necessitating focused attention. Table 2 below provides an overview of the challenges encountered by students when solving second-order NHDEs using the method of undetermined coefficients. Each challenge is subsequently explained through excerpts from the participants.

Table 2: Challenges faced by the participants

Method		Type of Difficulty	APOS Level	% age
Undetermined Coefficients	1.	Finding the **complementary** solution		
	(a)	Formation of the characteristic equation	action	
	(b)	Interiorisation of the above actions to the process of solving the characteristic equation to find the roots	process	
	(c)	Encapsulation of the above process to form the object of the general solution for the homogeneous equation for distinct, repeated and complex roots.	object	
	2.	Finding a **particular solution for** the NHDEs:		
	(a)	Formation of the initial guess	action	
	(b)	Finding the derivative of the initial guess	process	
	(c)	Substituting the derivatives of the initial guesses into the main equation to find the undetermined coefficients for the particular solution.	process	
	(d)	Finding a particular solution for NHDEs	object	
	3.	Organisation of the above actions, processes and objects in a coherent schema, for students to be able to deal with any problems concerning the NHDEs, that is, ability to write the general solution for the NHDEs.	schema	

Formation of the characteristic equation

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The majority of students demonstrated proficiency in forming and solving the characteristic equation/complementary function of a homogeneous equation. A small number of students who encountered difficulties in forming characteristic equations from homogeneous equations made minor errors or slips. These slips are typically attributed to memory deficits rather than fundamental misunderstandings, and are more easily identifiable than conceptual errors (Russell & Masters, 2009). However, a minute percentage of the participants indicated a lack of complete understanding in obtaining a characteristic equation. For instance, student ST8 provided the following answer:

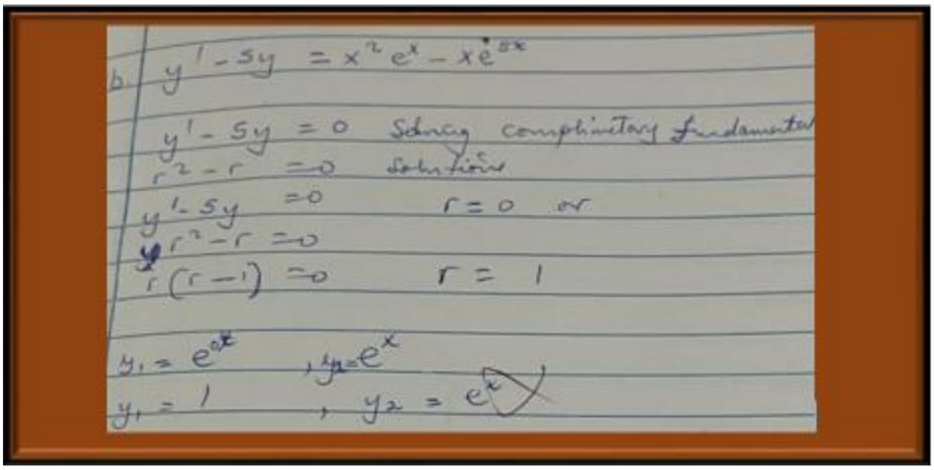

Figure 2: Wrong Characteristic

The origin or cause of this error could hardly be determined since the auxiliary equation written is not in any way related to the homogeneous equation. This shows that the participant had insufficient mastery of basic facts, concepts and skills in homogeneous equations, which was closely connected with limitations of imagination and creativity in new situations (Legutko, 2008).

There is also a group of students who had no problems with forming the characteristic equation but could not follow the actual procedures required to find the equation. Most of them seemed to memorise that y'' stands for m^2, y' for m^1 and y for $m^0 = 1$, without understanding where this was coming from. For example, given $y'' + 4y' + 3y = 0$, making use of the arbitrary value of $y(x) = e^{mx}$ to find the characteristic/auxiliary equation, the process goes as follows:

Let $y(x) = e^{mx}$, then $y'(x) = m\, e^{mx}$, $y'' = m^2e^{mx}$,

Substituting in the ODE, we have: $m^2e^{mx} + 4\, m\, e^{mx} + 3\, e^{mx} = 0$.

Dividing throughout by e^{mx}, we remain with $m^2 + 4\, m + 3 = 0$.

The participants' performance in this case aligns with the action stage of the APOS theory. According to Dubinsky et al. (2005), students in the *action* stage can identify variations in a pattern but may struggle to formulate a rule that generates the pattern. Thus, the participants demonstrated attributes consistent with the *action* stage of the APOS theory as they endeavored to derive the characteristic equation of a homogeneous equation by following a specific pattern whose development they did not fully comprehend. Rasmussen and Whitehead (2003) have noted instances where students arrived at correct conclusions in NHDEs, yet harbored misconceptions and conceptual gaps. However, the absence of a mandatory

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requirement to use e^{mx} or demonstrate the formation of a characteristic equation made it challenging to discern between those who comprehended the concept and those who did not. Discussions with a few student teachers revealed a tendency to rely on memorisation. Some participants expressed that they could easily derive the characteristic equation without fully understanding why it should be structured in a particular manner. This form of rote learning poses a risk of inhibiting active learning.

Finding the General Solution of a homogeneous equation

On finding the solution for the characteristic equation, participants showed that they had a good grasp of writing the general solution for a homogeneous equation with **distinct roots**, for example, given that $\mathbf{m^2 -3m - 4 = 0,}$ then m = 4 or -1, hence the general or complementary solution for the homogeneous equation becomes $y_c = C_1e^{4t} + C_2e^{-t}$. The researchers identified a few errors, which did not raise concerns due to the overall satisfactory performance in solving a simple quadratic equation displayed by the students. Notably, the participants' proficiency exceeded the process stage of the APOS theory and reached the object stage, as evidenced by their ability to determine the general solution for the homogeneous equation comprehensively. This indicates that the participants were able to encapsulate the process of calculating the distinct roots of the characteristic equation to form the object of the general solution for the homogeneous equation.

The participants could find the set of distinct roots as a process upon which they could find the general solution for the homogeneous equation as an entity (object). The roots were found as a basis to establish the complementary solution. According to Dubinsky and McDonald (2001), students at the object level can present solutions using different representations (processes) and articulate the reasons behind the choice of a method for solving a given problem. However, the researchers expressed concern about specific errors made by a subset of students, including both low and high achievers. For instance, ST1 in question A4 submitted the following solution:

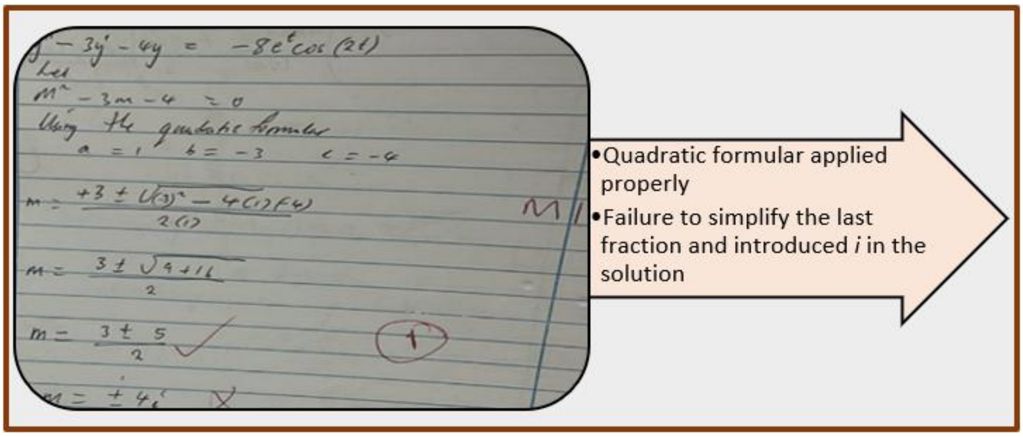

Figure 3: Misconception of complex numbers

In simplifying $m = \frac{3\pm5}{2}$, the answer was given as $\pm4i$. The concerning issue in this response lies in the participant's method of obtaining ±4i as the solution from the square root of a positive number. This reveals a fundamental lack of understanding of basic complex number concepts. The participant struggles to

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interiorise the above action (formation of the characteristic equation) to the process of solving the characteristic equation using the quadratic formula to find the distinct roots. Difficulty in simplifying simple algebraic expressions was consistently observed among some in-service teachers. Although only a few students displayed weak algebraic skills, it was particularly evident when participants were tasked with finding the roots of a quadratic equation with distinct solutions. Overall, a significant number of participants successfully reached the *object* stage of finding solutions for a given equation with distinct roots, while a few remained stuck in the *process* stage.

Figures 4 and 5 also exhibit the comprehension errors made by the participants, which were exacerbated by weak algebraic skills.
ST5 in question B5 made two major errors:

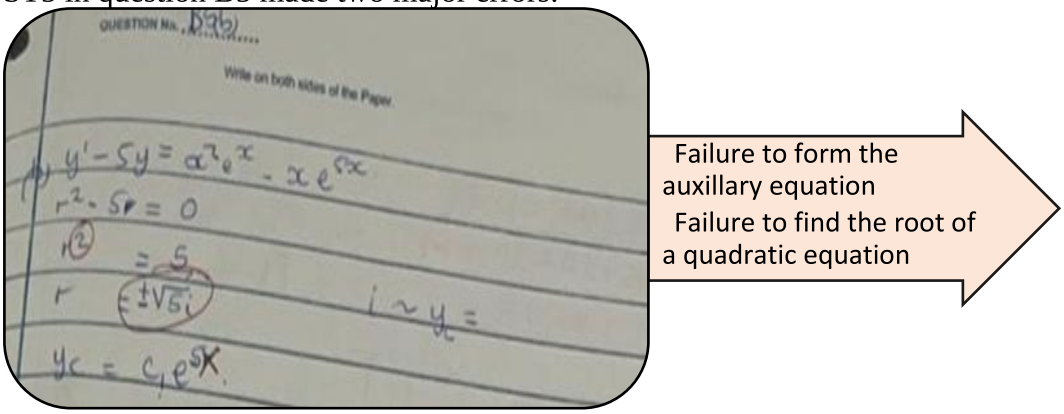

Figure 4: Failure to solve quadratic equation

ST5 could have made the first mistake because several times the differential equations started with y'' hence the auxiliary equation was wrong. From the homogeneous equation y' -5y =0, the auxiliary equation was written as r^2 -5r = 0 instead of r – 5 = 0. The second error involved solving r^2 = 5, to get r = $\pm\sqrt{5}i$. This error reflects another challenge of knowledge deficiency in complex numbers. The two errors contributed to failure to get the general solution for the homogeneous equation. ST21 also wrote:

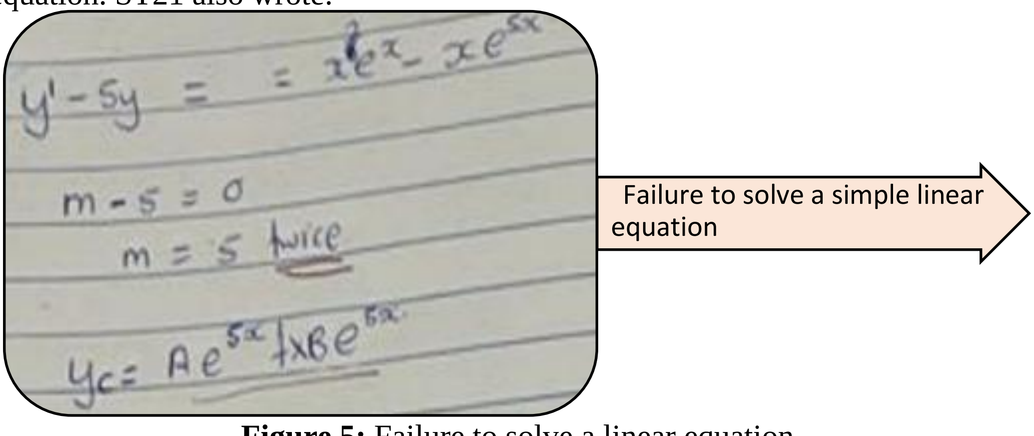

Figure 5: Failure to solve a linear equation

ST21 failed to write the correct solution for m-5 =0; hence, the general solution for the entire homogeneous equation was wrong. ST43 had this to write:

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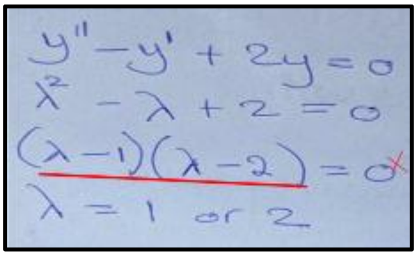

Figure 6: Lack of basic facts of factorisation

The inability of participant ST43 to solve the quadratic equation with complex roots and instead factorise the expression raises concerns about the mastery of pre-calculus skills, particularly in algebra. This observation aligns with Makamure and Jojo's (2021) and Makonye's (2016), studies, which highlighted the correlation between poor performance in differential calculus and inadequate pre-calculus skills. It is crucial to note that incorrect roots not only affect the solution of the non-homogeneous differential equations (NHDEs) but also render the complementary equation and the final NHDEs solution incorrect.

Furthermore, while the majority of participants demonstrated proficiency in formulating the general equation of a homogeneous equation with **repeated roots**, a small number of individuals made errors by omitting the function of x in the second root of the equation. An illustrative example is participant ST67's erroneous representation:

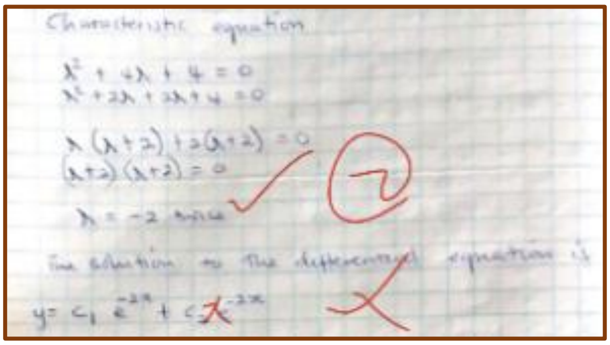

Figure 7: Misconceptions of NHDEs with repeated roots.

The omission of the variable x in the second root of the non-homogeneous equation by the participant has significant implications. Although seemingly minor, this omission can substantially impact the solution of the entire non-homogeneous equation. It appears that certain students were able to interiorise and encapsulate the action and process of forming and solving the characteristic equation respectively whilst failing to get to the object stage of transforming the action and process into an entity/object of writing the solution for the homogeneous equation (H/E).

Notably, the challenges were most pronounced when dealing with homogeneous equations featuring **complex roots**. It was anticipated that participants would have the requisite skills to address such equations; however, the actual demonstration revealed the opposite. When participants learnt about H/E, they should have grasped the concept that, if the roots of the characteristic equation happen to be $m_{1,2} = \lambda \pm \mu i$, then the general solution for the H/E becomes: $\mathbf{y(t) = C_1 e^{\lambda t} \cos(\mu t) + C_2 e^{\lambda t} \sin(\mu t)}$.

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However, most students just memorised the formula from lecture discussions without understanding how it is derived from Euler's formula:

$e^{i\theta}$= **cos θ + isin θ**. Also,

$e^{-i\theta}$ **= cos(−θ) + isin(−θ)** = cos θ **−** ***i*****sin θ**.

Hence, y_1 (t) = $e^{\lambda+\mu i}$ = $e^{\lambda t}$ (cos(μt) + isin(μt)),

y_2 (t) = $e^{\lambda-\mu i}$ = $e^{\lambda t}$ (cos(μt) − *i*sin(μt))

Finding $y_1(t) + y_2(t)$ and $y_1(t) - y_2(t)$ generates **y(t) = C₁** $e^{\lambda t}$ **cos (μt) +** $C_2 e^{\lambda t}$ **sin(μt).** (Zill, 2018). Failure to understand the formula resulted in most students omitting some variables in the formula and sometimes writing the formula terms wrongly. The following example from the excerpts highlights some of the misconceptions:

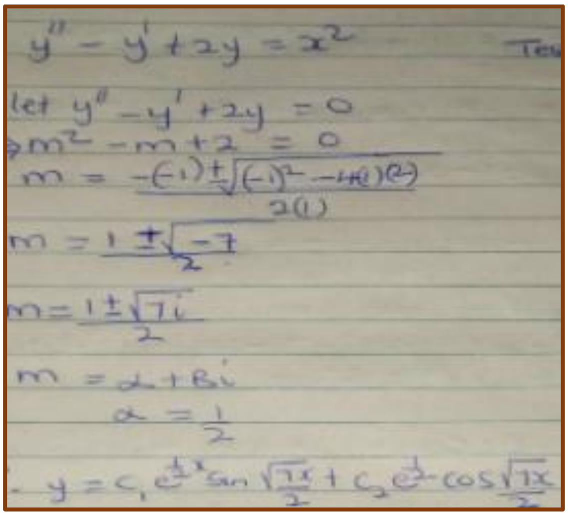
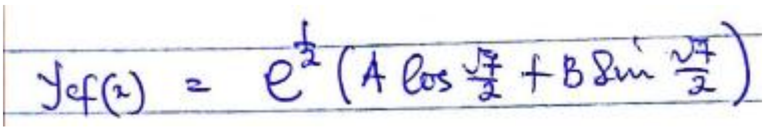


Figure 8: Misconceptions of complex roots

Despite the fact that the formation and solution for the characteristic equation (action and process, respectively) are correct, the complementary solution y_c is wrong with the variable **x** written inside the square root sign. The same mistake was done twice in the first and second terms of the complementary solution y_c. This shows that the student does not know why the variable x appears in the formula. Instead of writing $\frac{\sqrt{7}}{2}$ x, the student wrote $\frac{\sqrt{7x}}{2}$ which is a totally different answer. This participant is failing to identify how the formula was obtained as well as failing to find variations in the pattern of the formula; hence, does not qualify to be in the 'object' stage of the APOS theory regarding capability to solve NHDEs with complex roots. In the second example, the candidate left out the variable ***x*** as follows:



The participant wrote cos $\frac{\sqrt{7}}{2}$ and sin $\frac{\sqrt{7}}{2}$ only without the variable ***x***.

The identified errors were quite obtrusive and were of a conceptual nature and significantly impeded the resolution of the NHDEs. According to Legutko (2008), conceptual errors stem from a lack of knowledge and are closely associated with limitations in imaginative and creative thinking in novel situations. The

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performance of the research subjects indicated an inadequate mastery of NHDEs with complex roots. These issues could prove to be detrimental to students' learning, particularly when a topic is intended to provide foundational information for subsequent modules.

Finding a particular solution for the NHDEs

The initial step in solving NHDEs, as highlighted in the preceding sections, is to solve the corresponding H/G equation to find y_c. The second stage is to find the particular solution y_p so that the general solution for the NHDEs is $y(t) = y_c(t) + y_p(t)$. Finding the particular solution for a NHDEs involves several stages and different participants had difficulties at various stages. In this case, some errors were individualistic and idiosyncratic whilst some were common across all levels of the participants. The first stage involved formation of initial guesses, which was the action stage of the APOS theory. Finding the initial guesses was not much of a challenge when the right hand side (RHS) of the NHDEs was a polynomial. For example, $y'' - y' + 2y = \boldsymbol{x^2}$. In this case most participants could easily write the initial guess $y_p = Ax^2 + Bx + C$, with A,B,C being the undetermined coefficients.

The two processes which involved finding the corresponding derivatives and substitution into the main equation to find the numerical values of the undetermined coefficients A,B,C for polynomials were also manageable for more than 90% of the participants. However, the major problems arose where the initial guesses involved trigonometrical ratios and exponentials. Chikwanha (2021) concurs that the problems in solving differential equations emanate from the misconceptions found in topics like the power rule, chain rule and exponentials. The following excerpts validate the challenges faced by the participants on question 2 of the test:

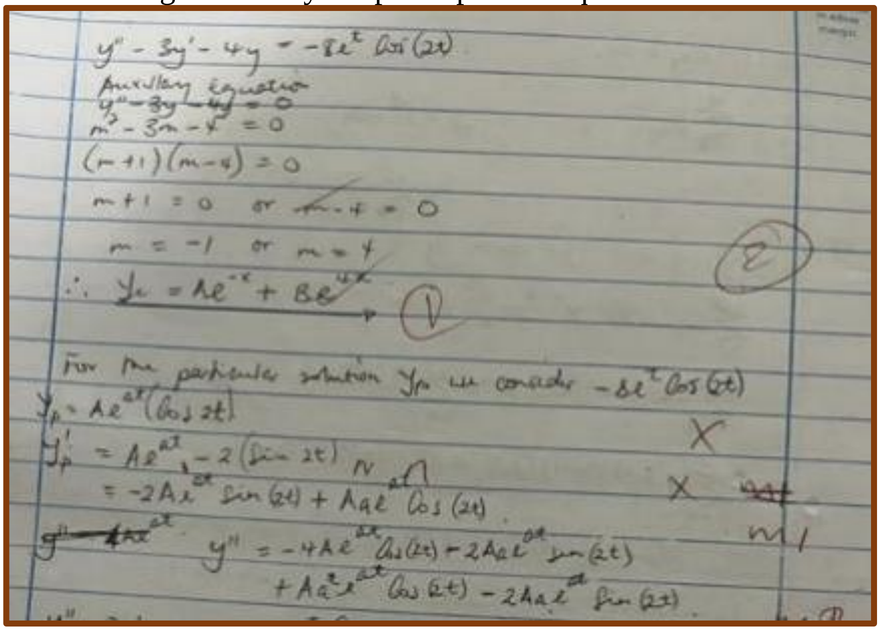

Figure 9: Differentiation challenges

The individual in question encountered various obstacles despite being capable of identifying the complementary solution. First, the participant had no idea of how the initial guess of the combination of exponents and trigs is formed. According to Arnon et al. (2014), the development of any mathematical concept

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commences with an action, a sentiment echoed by Dubinsky and McDonald (2001) who assert that the inception of every mathematical concept occurs through an action in the learner's mind. This action must be interiorised to form a process; hence, the absence of the initial action impedes the development of an entity into a process.. In this instance, two processes needed execution. The participant's inability to devise a suitable initial guess bred the second problem. For example the participant struggled to determine the derivative y′p of the "incorrect" initial guess, and faced further challenges with substituting the derivatives into the main equation. This pattern of errors persisted as an erroneous y′p led to an erroneous y″p, resulting in incorrect derivative substitution and ultimately, an incorrect particular solution. However, the focus of the researcher was on addressing the concepts that eluded the participants, namely the incapacity to formulate an initial guess for a nonhomogeneous differential equation with exponents and trigonometric functions (action), the inability to differentiate exponents and trigonometric functions in the initial guess, and the challenges associated with derivative substitution (processes). Consequently, the particular solution (object) proved to be inaccurate. Additional example responses to the same question are provided below:

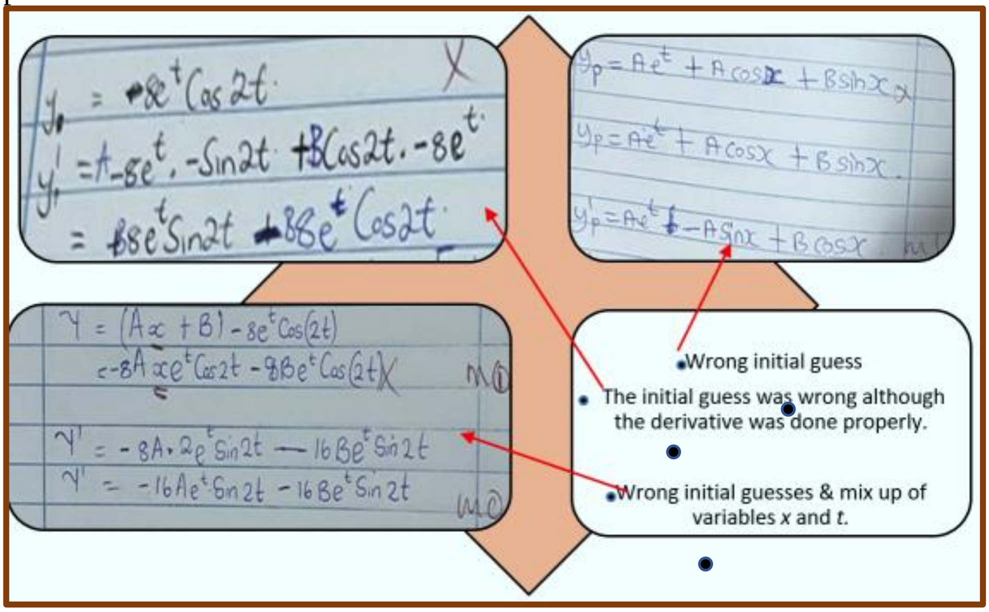

Figure 10: Excerpts of challenges in NHDEs

By and large, the participants had varied initial guesses for the same question, which were mostly wrong. About 60% of the participants got the wrong initial guess for this question. The same trend was observed in question 3, that is, $y' - 5y = x^2e^x - xe^{5x}$. A significant number (68%) of the participants got the initial guess wrong for this particular question. Most of them wrote the initial guess as $e^x (Ax^2 + Bx + C) + e^{5x} (Dx + E)$ instead of $e^x (Ax^2 + Bx + C) + \mathbf{x}\, e^{5x} (Dx + E)$. The participants failed to realise that the second root of the initial guess had overlapping terms with the general solution $y_c = c_1e^{5x}$. Hence, to avoid these overlapping terms, the second root of the initial guess had to be multiplied by the variable **x.** Generally, this study revealed that many participants encountered challenges when dealing with particular solutions for non-homogeneous differential equations (NHDEs) that involved exponents and trigonometric functions. Herholdt and Sapire (2014)

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suggested that a prevalence of certain types of errors may indicate a general confusion, highlighting the necessity for further clarification of the underlying concepts. Despite this confusion, some participants were able to make correct initial guesses on the problem. However, they struggled with determining the derivatives of these initial guesses, indicating a lack of fundamental understanding of the product rule for derivatives, which states that the derivative of the product of two functions f(x) and g(x) is given by $\frac{dy}{dx}$ f(x)·g(x) = [g(x) × f'(x) + f(x) × g'(x)]. According to Chikwanha (2021), educators and students encounter difficulties in differentiation and integration, which are foundational concepts for Ordinary Differential Equations (ODEs). These challenges may have impacted the academic performance of participants in Non-Homogeneous Differential Equations (NHDEs) due to the calculus-intensive nature of the topic (Makamure & Jojo, 2022). Ningsih and Rohana (2018) posit that a comprehensive grasp of fundamental concepts such as logarithms, derivatives, and integral functions can facilitate the resolution of ODEs. Furthermore, these scholars assert that students face both conceptual and procedural obstacles in solving second-order differential equations, stemming from inadequate comprehension of essential calculus principles.

Following the identification of the correct initial guess and the corresponding values of yp, y′p, and y″p, many participants consistently encountered challenges in substituting these values into the main equation and simplifying the resulting algebraic expressions to determine the undetermined coefficients. A notable number of participants demonstrated inaccuracies in performing algebraic manipulations, particularly when simplifying and equating corresponding values on both sides of the equation. As a result, determining the numerical values of the undetermined coefficients proved to be a formidable task for the participants. The following excerpts from the participants' work serve as examples of the errors in substitution that were observed:

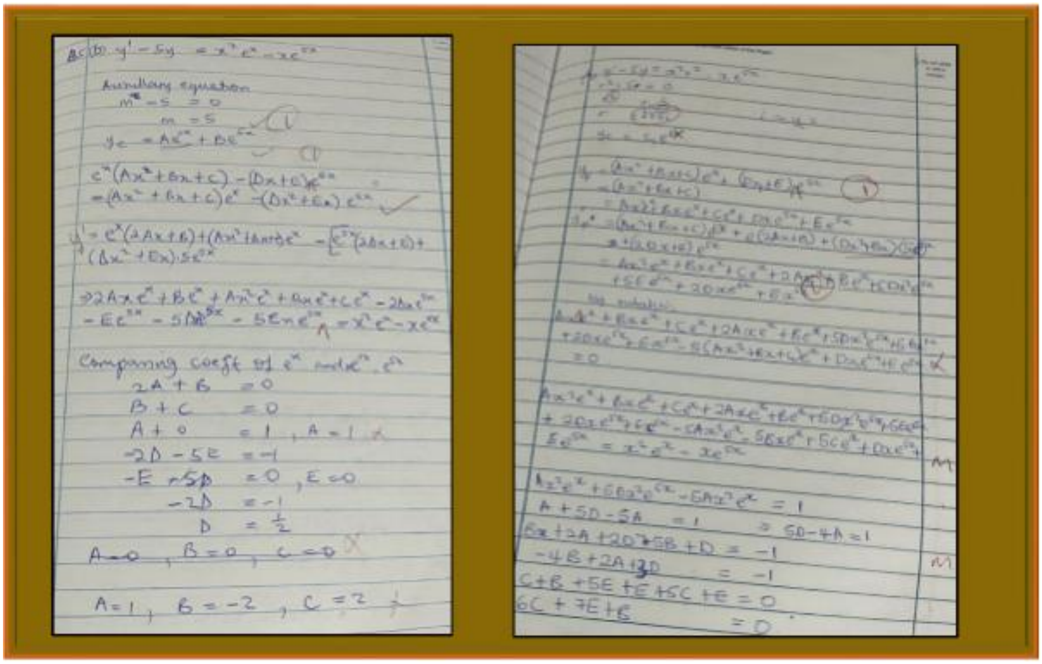

Figure 11: Substitution Errors

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Sometimes the participants failed to include the variable **x** in the initial guess whose terms overlapped with the general solution y_c. This way, they could only determine the values of the coefficients A,B,C and could not create equations that would enable them to find the values of D and E because the overlapping terms cancelled each other. As a result, it affected the values of the undetermined coefficients and, consequently, the general solution y(x) of the NHDEs. The extract below clarifies this explanation from the example question $y' - 5y = x^2e^x - xe^5$:

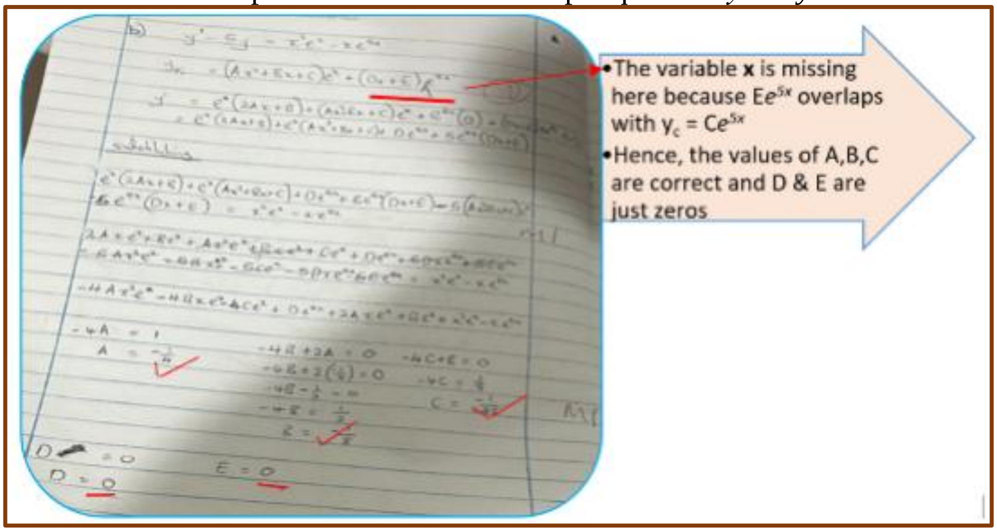

Figure 12: Challenges with repeated roots

Generally, participants encountered challenges in determining the initial guesses and subsequent particular solutions for Nonlinear Homogeneous Differential Equations (NHDEs). The following excerpts from participants' responses regarding the equation $y' - 5y = x^2e^x - xe^{5x}$ illustrate the difficulties faced:

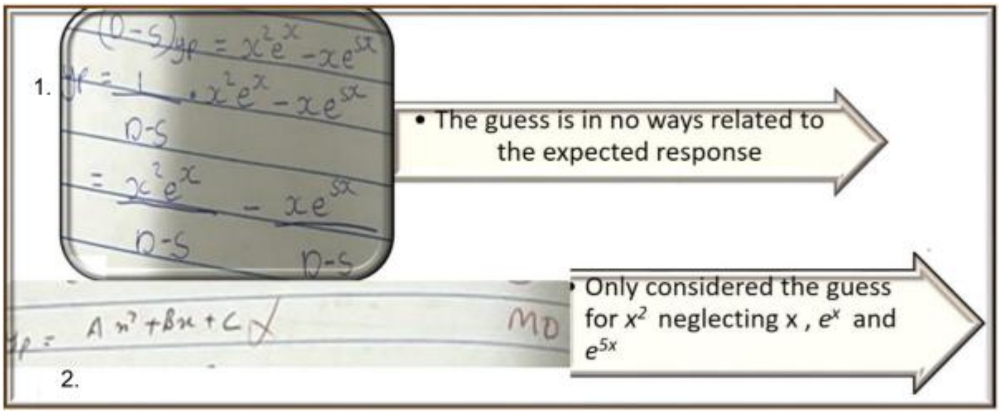

Figure 13: Wrong Guesses

In the context of questions involving exponents, sines, and cosines, it was observed that only a minority of participants (less than 40%) were able to reach the object stage of the APOS theory. The participants encountered difficulties in organizing the actions, processes, and objects into a cohesive schema necessary for addressing problems related to NHDEs. The challenges faced by the research participants align with the findings of Yarman et al. (2020), indicating that students' proficiency in differential equations is relatively low. Particularly in the case of NHDEs, the limited performance may have been exacerbated by the fact that most

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students are accustomed to comprehending solutions to equations in the form of numerical values or concise algebraic functions rather than lengthy functions (Ningsih & Mulbasani, 2019; Prawoto et al., 2018). Ningsih and Rohana (2018) emphasize the importance of students grasping the concept of the complementary function and the particular solution to effectively address NHDEs.

CONCLUSION

The analysis of the test participants' responses revealed significant inadequacies in their performance. Although these inadequacies were individualised, the research participants seldom reached the schema stage of the APOS theory. Specifically, the student teachers exhibited a limited understanding of NHDEs concepts and encountered substantial challenges stemming from their misconceptions of fundamental calculus concepts. Consequently, their inability to grasp certain concepts propelled them to memorise formulas without comprehension. Prawoto et al. (2018) corroborate that students' difficulties in solving second-order NHDEs problems often stem from a lack of both conceptual and procedural understanding, resulting in consistently poor performance in this area. Furthermore, a deficient understanding of basic calculus concepts may impel participants to prioritise rote methods over conceptual comprehension of NHDEs, which can be transferred to the learners they teach. Mudrikah (2016) thus advocates for a mathematics education aimed at fostering competencies that enable students to communicate, solve problems, reason logically, make conjectures, and ultimately cultivate a positive attitude towards mathematics. This assertion recognises mathematics not merely as a fixed body of knowledge and skills to be memorised (Conway & Sloane, 2005).

However, most of the challenges that the participants had in solving NHDEs originated from their failure to understand basic and pre-requisite topics that were feeder topics into NHDEs. Some of these basic skills are:

(i) Lack of basic knowledge of the product rule for derivatives $\frac{dy}{dx}\mathbf{f(x)\cdot g(x) = [g(x) \times f'(x) + f(x) \times g'(x)]}$, or basic principles of calculus
(ii) A significant number exhibited miscalculations errors in manipulating algebraic operations, that is, failure to substitute
(iii) Failure to solve simple quadratic equations; hence, complementary equations were wrong
(iv) Misconceptions in computing complex numbers
(v) In addition, learning by memorisation contributed to their challenges.

By and large, the performance of participants in NHDEs was predominantly affected by comprehension errors. These errors included inadequate algebraic skills, thin knowledge of differentiation rules, difficulties in solving simple linear and quadratic equations (especially evident in solving the auxiliary/characteristic equations of homogeneous equations), improper use of formulas, technical errors, and miscalculations. This suggests that most students had a basic understanding of NHDEs, with a few at the *process* level, and even fewer at the *object* level. Only a handful were able to reach the *schema* level, indicating their ability to organise actions, processes, and objects into a coherent schema to address NHDE problems.

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Recommendations

The results of this study highlight the necessity of providing participants with pre-calculus skills before introducing NHDEs. The lack of understanding of fundamental concepts such as algebra and pre-calculus among participants indicates deficiencies in their knowledge of mathematics teaching, as these concepts are expected to be taught to the learners they instruct. It can therefore be concluded that enhancing student teachers' performance and retention in NHDEs classes requires individualistic and innovative instructional approaches to enhance understanding.

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